ON THE STABILITY OF RANDOM PROCESSES WITH DISTRIBUTED PARAMETERS

(OB USTOICHIVOSTI SLUCHAINYKH PROTSESSOV S RASPREDELENNYMI PARAMETRAMI)

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In this paper we consider the probable stability of random processes described by partial differential equations. Theorems are proved on probable stability which are analogous to the theorems of Liapunov's second method.

The stability of the solution of a system of ordinary equations with random parameters was considered by Kats and Krasovskii [1].

1. Let us consider perturbed random processes described by the system of partial differential equations

$$\frac{\partial \varphi_i}{\partial t} = f_i\left(t, x, y, z, \varphi_s, \frac{\partial \varphi_s}{\partial x}, \frac{\partial \varphi_s}{\partial y}, \frac{\partial \varphi_s}{\partial z}, \frac{\partial^2 \varphi_s}{\partial x^2}, \ldots, \frac{\partial^2 \varphi_s}{\partial z^2}, u_1, \ldots, u_\alpha\right) \qquad (i, s = 1, \ldots, n) \tag{1.1}$$

Here $\varphi_i = \varphi_i(t, x, y, z)$ are functions characterizing the state of the process, x, y, z are the coordinates of a region τ in which the process runs, t is the time, $u_q = u_q(t)$ $(q = 1, \ldots, \alpha)$ are random parameters and

$$\{u_1, \ldots, u_\alpha\} \in U.$$

For example, when studying the flow of a liquid or of a gas such parameters may be the coefficient of viscosity, the density of the free stream, and other such quantities.

It is assumed that

$$f_i(t, x, y, z, 0, \ldots, 0, u_1, \ldots, u_\alpha) = 0$$

$$\text{when} \quad t \geqslant t_0, \qquad (x, y, z) \in \tau, \quad \{u_1, u_2, \ldots, u_\alpha\} \in U$$

and the system (1.1) has a solution for the prescribed initial and boundary conditions and for realizations $u_q{}^p = u_q{}^p(t)$ $(q = 1, \ldots, \alpha)$. The solution $\varphi_i \equiv 0$, $t \geqslant t_0$ $(i = 1, \ldots, n)$ corresponds to the unperturbed motion. The perturbed motion differs from the unperturbed in that it has different initial conditions. Let us denote $\varphi \equiv (\varphi_1, \varphi_2, \ldots \varphi_r)$ and $u \equiv (u_1, u_2, \ldots, u_\alpha)$.

The realizations of the random parameter $u^p(t)$ may have discontinuities. For example, $u^p = U_1$ may hold for $t_0 \leqslant t \leqslant t_1$ and $u^p = U_2$ for $t_1 \leqslant t \leqslant t_2$, etc., where the quantities $U_1, U_2, \ldots$ are such that when they are substituted into system (1.1) that latter has a solution with respect to φ_i $(i = 1, \ldots, n)$. At the instant of discontinuity of the parameter $u^p(t)$ the functions f_i and, consequently, the derivatives $\partial\varphi_i/\partial t$ $(i = 1, \ldots, n)$ also have discontinuities, while the solutions φ_i $(i = 1, \ldots, n)$ remain continuous vector-functions. For example, let the coefficient of viscosity ν of a certain liquid take the two values ν_1 and ν_2 with specified probabilities. At the instant $t = t_1$ when passing from the state ν_1 to ν_2 the velocity and pressure distribution of the liquid remain continuous functions. The distributions obtained for $\nu = \nu_1$ at the instant $t = t_1$ are the initial conditions for the liquid flow when the coefficient of viscosity is $\nu = \nu_2$.

The random vector-function

$$\{\varphi\,(\varphi_0,\, u_0,\, t_0;\, t,\, x,\, y,\, z),\ u\,(u_0,\, t_0;\, t)\} \equiv \{\varphi,\, u\}$$

whose realization satisfies the system of equations (1.1) will be called the solution of system (1.1).

At each instant t let us define the metrics (distances) $\rho_0 = \rho_0[\varphi, u, t]$ and $\rho = \rho[\varphi, u, t]$ which are real nonnegative numbers for any solution of system (1.1) in region τ and are such that $\rho_0[0, u, t] \equiv 0$ and $\rho[0, u, t] \equiv 0$.

The initial state when $t = t_0$ will be characterized by the metric ρ_0 and the state at an arbitrary instant $t \geqslant t_0$, by the metric ρ.

We shall consider solutions of system (1.1) which satisfy the condition $\rho_0 < H_0$ when $t = t_0$, where H_0 is a positive constant.

The metric $\rho = \rho[\varphi, u, t]$ is said to be continuous in the metric $\rho_0[\varphi, u, t]$ if for any number $\varepsilon > 0$ at $t = t_0$ we can find a number $\delta = \delta(\varepsilon) > 0$ such that the inequality $\rho < \varepsilon$ will be satisfied when $\rho_0 < \delta(\varepsilon)$ and $t = t_0$.

In what follows we shall take it that the metric $\rho = \rho[\varphi, u, t]$ is continuous in the metric $\rho_0 = \rho_0[\varphi, u, t]$ at $t = t_0$.

But we shall not assume the converse, i.e. the metric ρ_0 may not be continuous in the metric ρ. For example,

$$\rho = \left\{\int_\tau \sum_{i=1}^n \varphi_i^2\, d\tau\right\}^{1/2}, \qquad \rho_0 = \left\{\int_\tau \sum_{i=1}^n \left[\varphi_i^2 + \left(\frac{\partial\varphi_i}{\partial x}\right)^2 + \left(\frac{\partial\varphi_i}{\partial y}\right)^2 + \left(\frac{\partial\varphi_i}{\partial z}\right)^2\right] d\tau\right\}^{1/2}$$

In this case, for a given $\varepsilon > 0$ there exist a $\delta(\varepsilon) > 0$ such that $\rho < \varepsilon$ if $\rho_0 < \delta(\varepsilon)$.

Let us now introduce a certain functional $v = v[\varphi, u, t]$ which at a fixed instant t and for a given vector-function $\{\varphi(t, x, y, z), u(t)\}$ takes a real value.

We shall assume that when $\varphi = 0$, $v[0, u, t] = 0$. For example, ρ and ρ_0

are such functionals.

The functional $v = v[\varphi, u, t]$ is said to be positive (negative) definite in the metric ρ if $v[\varphi, u, t] \geqslant 0$ $(v[\varphi, u, t] \leqslant 0)$ when $t \geqslant t_0$, and if for any positive number ϵ we can find a positive number $\delta = \delta(\epsilon)$, depending only on ϵ, such that the inequality

$$v[\varphi, u, t] \geqslant \delta(\varepsilon) \ (v[\varphi, u, t] \leqslant -\delta(\varepsilon))$$

is satisfied when $\rho \geqslant \varepsilon$ and $t \geqslant t_0$.

The positive definiteness of the functional depends upon the boundary conditions imposed on $\varphi(t,x,y,z)$. For example, the functional

$$v_1 = \int_a^b \left(\varphi \frac{\partial \varphi}{\partial x} + \varphi^2\right) dx$$

is easily put in the form

$$v_1 = \frac{1}{2}[\varphi^2(b) - \varphi^2(a)] + \int_a^b \varphi^2 \, dx$$

If $\varphi(a) = 0$, then the functional v_1 is positive definite in the metric

$$\rho = \left\{\int_a^b \varphi^2 \, dx\right\}^{1/2}$$

However, if $\varphi(a)$ can take arbitrary values, then v_1 is not positive definite.

The functional $v = v[\varphi, u, t]$ is said to be continuous in the metric $\rho_0 = \rho_0[\varphi, u, t]$ at $t = t_0$ if for any arbitrary small number $\epsilon > 0$ we can find a positive number $\delta = \delta(\epsilon)$ such that the estimate $|v| < \epsilon$ is satisfied when $\rho_0 < \delta(\epsilon)$ and $t = t_0$.

For a fixed instant t the functional $v = v[\varphi, u, t]$ takes random numerical values.

The mathematical expectation of the functional v at time $t \geqslant t_0$ under the condition that $\{\varphi, u\}$ is a solution of system (1.1) generated by the initial distribution

$$\{\varphi_0 = \varphi(t_0, x, y, z), \ u_0 = u^p(t_0)\}$$

is denoted by

$$M_t[v] = M[v[\varphi, u, t]; \varphi, u, t/\varphi_0, u_0, t_0]$$

Further, we introduce the notations

$$M_t[v < \varepsilon] = \int_{V<\varepsilon} V dF(V), \qquad M_t[v \geqslant \varepsilon] = \int_{V \geqslant \varepsilon} V dF(V)$$

where $F(V) = p(v < V)$ is the probability distribution function of the random variable v, so that we shall have

$$M_t[v] = M_t[v < \varepsilon] + M_t[v \geqslant \varepsilon]$$

Here $u_0 = u^p(t_0)$ is an actual realization of $u(t)$ at the instant $t = t_0$ and, here, the functional v takes on the actual numerical value

$v_0 = v[\varphi_0, u_0, t_0]$ and, therefore,

$$M_{t_0}[v] = M[v[\varphi_0, u_0, t_0]; \varphi_0, u_0, t_0 / \varphi_0, u_0, t_0] = v[\varphi_0, u_0, t_0]$$

We now define stability.

The unperturbed process $\varphi \equiv 0$ is said to be probably stable in the metrics ρ and ρ_0 if for any arbitrarily small numbers $\epsilon > 0$ and p $(0 < p < 1)$ we can find a number $\delta = \delta(\epsilon, p) > 0$ such that for every solution of system (1.1) which at the initial instant $t = t_0$ satisfies the inequality $\rho_0 < \delta(\epsilon, p)$, the inequality $p_t(\rho < \epsilon) > 1 - p$ will be satisfied for all $t \geqslant t_0$.

Here $p_t(\rho < \epsilon)$ is the probability that at the instant t the inequality $\rho < \epsilon$ is satisfied.

If probable stability in the metrics ρ and ρ_0 holds and, moreover, if for any $\gamma > 0$ the condition

$$\lim p_t(\rho < \gamma) = 1 \quad \text{as} \quad t \to \infty$$

is fulfilled for all solutions with initial conditions satisfying the inequality

$$\rho_0 < H_1 \tag{1.2}$$

where H_1 is a positive constant, then this unperturbed process $\varphi \equiv 0$ is said to be probably asymptotically stable in the metrics ρ and ρ_0, and region (1.2) lies in the region of attraction of the unperturbed motion.

Thus, when the process is probably asymptotically stable the probability that ρ is arbitrarily close to zero, equals unity as $t \to \infty$.

2. Below we prove theorems analogous to the stability theorems of Liapunov's second method.

T h e o r e m 2.1 . For probable stability in the metrics ρ and ρ_0 of the process $\varphi \equiv 0$ it is sufficient that there exists a functional $v = v[\varphi, u, t]$ which is positive definite in the metric ρ and continuous in the metric ρ_0 at $t = t_0$ and that the mathematical expectation $M_t[v]$ of this functional by virtue of the system (1.1) would not increase with time t.

P r o o f . Let there be given in advance two positive numbers ϵ and p $(0 < p < 1)$. Since the functional v is positive definite in the metric ρ, for the given $\epsilon > 0$ we can find a positive number $\epsilon_1 = \epsilon_1(\epsilon)$ such that $v \geqslant \epsilon_1(\epsilon)$ for any value of $\rho \geqslant \epsilon$ for any $t \geqslant t_0$ otherwise, if $v < \epsilon_1(\epsilon)$, then $\rho < \epsilon$. We choose the number $\delta > 0$ in the following way.

a) The metric ρ is continuous in the metric ρ_0 at the instant $t = t_0$, i.e. for given $\epsilon > 0$ we can find number $\delta_1 = \delta_1(\epsilon) > 0$ such that the inequality $\rho < \epsilon$ is fulfilled if ρ_0 satisfies the condition $\rho_0 < \delta_1(\epsilon)$ at the instant $t = t_0$.

b) The functional v is continuous in ρ_0 at the instant $t = t_0$, i.e. for any given $\epsilon_2 > 0$ we can find a number $\delta_2 = \delta_2(\epsilon_2) > 0$ such that the estimate $v < \epsilon_2$ is fulfilled if ρ_0 satisfies the condition $\rho_0 < \delta_2(\epsilon_2)$ at the instant $t = t_0$.

The number ϵ_2 is chosen to equal $\epsilon_2 = p\epsilon_1(\epsilon) = \epsilon_2(\epsilon, p)$. Let $\delta = \delta(\epsilon, p) = \min(\delta_1, \delta_2)$. Thus, for the prescribed $\epsilon > 0$ and $p > 0$ we can find a $\delta = \delta(\epsilon, p) > 0$, such that the inequalities $v < \epsilon_2(\epsilon, p)$ and $\rho < \epsilon$ are satisfied at the initial instant $t = t_0$ for every $\rho_0 < \delta(\epsilon, p)$.

We convince ourselves that if for the prescribed ϵ and p the quantity $\delta = \delta(\epsilon, p)$ is determined in the above manner and if at the initial instant

$t = t_0$ the solution $\{\varphi, u\}$ of system (1.1) satisfies the condition $\rho_0 < \delta(\varepsilon, p)$, then at any time $t \geq t_0$ the inequality $p_t(\rho < \varepsilon) > 1 - p$ is satisfied. But if $v < \varepsilon_1(\varepsilon)$, then $\rho < \varepsilon$. Consequently, the probability that $\rho < \varepsilon$ is not less than the probability $p_t(v < \varepsilon_1)$ i.e.

$$p_t(v < \varepsilon_1) \leqslant p_t(\rho < \varepsilon)$$

Therefore, to establish the probable stability it suffices to verify

$$p_t(v < \varepsilon_1) > 1 - p \quad \text{when } t \geqslant t_0$$

Let the initial conditions satisfy the inequality $\rho_0 < \delta(\varepsilon, p)$; consequently, $v < \varepsilon_2$ when $t = t_0$. The mathematical expectation of functional v is a nonincreasing variable, therefore,

$$M_t[v] \leqslant M_{t_0}[v] = v|_{t_0} \leqslant \varepsilon_2(\varepsilon, p) = p\,\varepsilon_1(\varepsilon) \tag{2.1}$$

Let us assume that when $t = T$ we have $p_T(v < \varepsilon_1) \leqslant 1 - p$; then $p_1 = p_T(v \geqslant \varepsilon_1)$, the probability that the realization will leave the region $v < \varepsilon_1$ at the instant $t = T$, is greater than or equal to p, i.e. $p_1 \geqslant p$.

Taking the inequalities $M_T[v < \varepsilon_1] \geqslant 0$ and $M_T[v \geqslant \varepsilon_1] \geqslant \varepsilon_1 p_1$ into account, we obtain the estimate

$$M_T[v] = M_T[v < \varepsilon_1] + M_T[v \geqslant \varepsilon_1] \geqslant M_T[v \geqslant \varepsilon_1] \geqslant \varepsilon_1 p_1 \geqslant p\varepsilon_1(\varepsilon) = \varepsilon_2 \tag{2.2}$$

which contradicts (2.1), the condition for the mathematical expectation not to increase. Consequently, we should have $p_t(v < \varepsilon_1) > 1-p$ and $p_t(\rho < \varepsilon) > 1-p$ when $t \geqslant t_0$. The theorem is proved.

T h e o r e m 2.2. For probable asymptotic stability in the metrics ρ and ρ_0 of the process $\varphi \equiv 0$, it is sufficient that there exist a functional $v = v[\varphi, u, t]$ which is continuous in the metric ρ_0 at $t = t_0$ and positive definite in the metric ρ and whose mathematical expectation does not increase with time by virtue of the system (1.1) and $\lim M_t[v] = 0$ as $t \to \infty$.

P r o o f . The conditions of Theorem 2.1 are fulfilled and, consequently, the solution $\varphi \equiv 0$ is probably stable in the metrics ρ and ρ_0. Let us verify that the solution $\varphi \equiv 0$ is probably asymptotically stable. For this, in addition to probable stability, it is necessary to check the equality $\lim p_t(\rho < \gamma) = 1$ as $t \to \infty$, where γ is an arbitrary small positive number.

We introduce the probability distribution function of the random variable v

$$F(V) = p_t(v < V)$$

for the instant t being considered.

Taking into account that v can take on only positive values, the mathematical expectation $M_t[v]$ is put in the form

$$M_t[v] = \int_0^\infty [1 - F(V)]\, dV$$

Here, the integrand is nonnegative and nonincreasing. According to the conditions of the theorem we have

$$\lim_{t\to\infty} M_t[v] = \lim_{t\to\infty} \int_0^\infty [1 - F(V)]\, dV = 0$$

Hence it follows that almost everywhere

$$\lim [1 - F(V)] = 0 \quad \text{or} \quad \lim F(V) = \lim p_t(v \leqslant V) = 1 \quad \text{as } t \to \infty$$

The functional v is positive definite, i.e. for any positive number γ

there exists another positive number $\delta(\gamma)$ such that when $\rho \geqslant \gamma$, $v \geqslant \delta(\gamma)$. Therefore, $p_t(v < \delta(\gamma)) \leqslant p_t(\rho < \gamma)$. But, $\lim p_t(v < \delta(\gamma)) = 1$ as $t \to \infty$. Thus, for any positive number γ we have $\lim p_t(\rho < \gamma) = 1$ as $t \to \infty$, i.e. the process $\varphi \equiv 0$ has probable asymptotic stability.

Note. We note that in the proofs of Theorems 2.1 and 2.2 the actual form of Equations (1.1) were not used. Therefore, Theorems 2.1 and 2.2 also apply for process as described by equations differing from (1.1), for example, the equations of liquid motion

$$\frac{\partial v_i}{\partial t} = -\sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} - \frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \sum_{j=1}^{3} \frac{\partial^2 v_i}{\partial x_j}, \qquad \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j} = 0 \qquad (i = 1, 2, 3)$$

where not all the equations contain time derivatives (v_i are the velocity components, ρ is the density, P is the pressure, x_j are the coordinates).

3. In this section we shall consider the random process $u(t,x,y,z)$ which is a homogeneous Markov process with a finite or infinite number of states. The limit

$$\frac{dM_t[v]}{dt} = \lim_{t\to t_1+0} \frac{M[v[\varphi, u, t]; \varphi, u, t / \varphi_1, u_1, t_1] - v[\varphi_1, u_1, t_1]}{t - t_1} \tag{3.1}$$

will be called the mean derivative of functional v by virtue of the system (1.1) at the point $\varphi|_{t=t_1} = \varphi_1$, $u|_{t=t_1} = u_1$, $t = t_1$.

In the case of the functional

$$v = v[\varphi, u, t] = \int_\tau w(\varphi, u, t)\, d\tau \tag{3.2}$$

where $w = w[\varphi, u, t]$ is some function of φ, u, t, the mean derivative can be written in the form

$$\frac{dM_t[v]}{dt} = \lim_{t\to t_1+0} \int_\tau \frac{M[w; \varphi, u, t / \varphi_1, u_1, t_1] - w(\varphi_1, u_1, t_1)}{t - t_1}\, d\tau \tag{3.3}$$

Let the right-hand side of system (1.1) depend only on one random parameter $u = u(t)$ which is a homogeneous Markov process with a finite number of states. At each instant t the function $u = u(t)$ can take one value u_i from the finite set $U(u_1, \ldots, u_n)$ and, moreover, the probability $p_{ij}(\Delta t)$ of the change of values $u_i \to u_j$ in time Δt satisfies the condition

$$p_{ij} = \alpha_{ij}\Delta t + o(\Delta t), \qquad p_{ii} = 1 - \sum_{j\neq i} \alpha_{ij}\Delta t + o(\Delta t) \tag{3.4}$$

where $o(\Delta t)$ denotes an infinitesimal quantity of an order of smallness higher than Δt. Then, the mean derivative (3.3) will be quivalent to

$$\frac{dM_t[v]}{dt} = \int_\tau \Big\{\frac{\partial w}{\partial t} + \sum_{k=1}^{n} \frac{\partial w}{\partial \varphi_k} f_k\Big(t, x, y, z, \varphi_s, \frac{\partial \varphi_s}{\partial x}, \ldots, u_i\Big) + \\ + \sum_{j\neq i} [w(\varphi, u_j, t) - w(\varphi, u_i, t)]\, \alpha_{ij}\Big\}\, d\tau \tag{3.5}$$

In the case $u = u(t)$ has an infinite number of states, the transition probability from the value $u = \alpha$ to the value $u \leqslant \beta$ in the time Δt is denoted by

$$\begin{aligned} p(u \leqslant \beta \neq \alpha,\ t < t_1 < t + \Delta t / u = \alpha, t) &= q(\alpha, \beta)\, \Delta t + o(\Delta t) \\ p(u = \alpha, t < t_1 < t + \Delta t / u = \alpha, t) &= 1 - q(\alpha)\, \Delta t + o(\Delta t) \end{aligned} \tag{3.6}$$

where
$$q(\alpha) = \int_{-\infty}^{\infty} q(\alpha, \beta)\, d\beta \tag{3.7}$$

Then, for the mean derivative (3.5) we shall have Formula

$$\frac{dM_t[v]}{dt} = \int_{\tau} \Big\{ \frac{\partial w}{\partial t} + \sum_{k=1}^{n} \frac{\partial w}{\partial \varphi_k} f_k\Big(t, x, y, z, \varphi_s, \frac{\partial \varphi_s}{\partial x}, \ldots, u_i\Big) + \\ + \int_{-\infty}^{\infty} w(\varphi, \beta, t) \frac{\partial q(\alpha, \beta)}{\partial \beta}\, d\beta - w(\varphi, \alpha, t)\, q(\alpha) \Big\} d\tau \tag{3.8}$$

The following Theorem 3.1 will be a corollary of Theorem 2.1 when u is a Markov process.

T h e o r e m 3.1 . If for the system of differential equations (1.1) it is possible to find a functional v which is continuous in the metric ρ_0 at $t = t_0$ and positive definite in the metric ρ and whose mean derivative

$$\frac{dM_t[v]}{dt} \quad \text{when } t \geqslant t_0$$

by virtue of these equations is a nonpositive quantity, then the solution $\varphi \equiv 0$ is probable stable in the metrics ρ and ρ_0 .

In the case of ordinary differential equations with random parameters Theorem 3.1 was proved in [1].

T h e o r e m 3.2 . If for the system of differential equations (1.1) it is possible to find a functional v which is continuous in the metric ρ_0 at $t = t_0$ and positive definite in the metric ρ and whose mean derivative satisfies the inequality

$$\frac{dM_t[v]}{dt} \leqslant -cv \tag{3.9}$$

where c is a positive constant, then the solution $\varphi \equiv 0$ of system (1.1) is probably asymptotically stable in the metrics ρ and ρ_0 .

P r o o f . The conditions of Theorem 3.1 are fulfilled and, consequently, the process $\varphi \equiv 0$ is probably stable in the metrics ρ and ρ_0 . In order to prove probable asymptotic stability it is sufficient to verify that $\lim M_t[v] = 0$ as $t \to \infty$.

We find the methematical expectation of Expression (3.9)

$$M\left[\frac{dM_t[v]}{dt};\ \varphi, u, t / \varphi_0, u_0, t_0\right] \leqslant -cM_t[v] \tag{3.10}$$

But the following Formula holds:

$$\lim_{t \to t_1+0} \frac{M_t[v] - M_{t_1}[v]}{t - t_1} = \frac{dM[v; \varphi_1, u_1, t_1 / \varphi_0, u_0, t_0]}{dt_1} = M_{t_1}\left[\frac{dM_{t_1}[v]}{dt_1}\right] \tag{3.11}$$

Integrating (3.10) from $t = t_0$ to $t = T$ and taking (3.11) into account, we get

$$M_{t_0}[v] - M_T[v] \geqslant c \int_0^T M_t[v]\, dt$$

The mathematical expectation $M_{t_0}[v]$ is positive and nonincreasing and, consequently, bounded. Whence follows the convergence of the integral on the right-hand side as $T \to \infty$. Then, as $T \to \infty$ the integrand should unboundedly decrease to zero,

$$\lim M_t[v] = 0 \quad \text{as } t \to \infty$$

Thus, the conditions of Theorem 2.2 are fulfilled. Consequently, the process $\varphi \equiv 0$ is probably asymptotically convergent.

4. Let us consider some examples. 1. Let a certain probabilistic process be described by Equations

$$\frac{\partial \varphi}{\partial t} = a(x, u)\frac{\partial \varphi}{\partial x} + b(x, u)\varphi \tag{4.1}$$

where $a(x,u)$, $b(x,u)$ are random functions each of which may have two states

$$a(x, u_1) = a_1(x), \quad a(x, u_2) = a_2(x), \quad b(x, u_1) = b_1(x), \quad b(x, u_2) = b_2(x)$$

Let the probability of the change of values $u_i \to u_j$ be represented as $p_{ij} = \alpha_{ij}\Delta t + o(\Delta t)$. We obtain the sufficient condition for probable stability in the metric

$$\rho \equiv \rho_0 \equiv \left\{\int_0^l \varphi^2 dx\right\}^{1/2}$$

of the solution $\varphi \equiv 0$ of the system in the segment $[0, l]$.

We introduce the functional

$$v = \int_0^l \frac{1}{2} Q(x, u)\varphi^2 dx$$

which is positive definite in the metric ρ

The quantity $Q(x, u) \geqslant \varepsilon^0 > 0$ is random function. We denote

$$Q(x, u_1) = Q_1(x), \quad Q(x, u_2) = Q_2(x).$$

For probable stability the mean derivative

$$\frac{dM_t[v]}{dt} = \frac{1}{2}[a_i(l)\, Q_i(l)\,\varphi^2(l) - a_i(0)\, Q_i(0)\,\varphi^2(0)] - \int_0^l A_i(x)\varphi^2 dx$$

where

$$A_i(x) = \frac{1}{2}\frac{\partial a_i(x)\, Q_i(x)}{\partial x} - b_i(x)\, Q_i(x) - \frac{1}{2}\alpha_{ij}[Q_j(x) - Q_i(x)] \quad (j \neq i)$$

should be nonpositive when $i, j = 1, 2$.

The sufficient conditions for probable stability are

$$a_i(l) \leqslant 0, \qquad a_i(0) \geqslant 0, \qquad A_i(x) \geqslant 0 \qquad (i = 1, 2) \tag{4.2}$$

when the conditions

$$a_i(l) \leqslant 0, \qquad a_i(0) \geqslant 0, \qquad A_i(x) \geqslant \varepsilon > 0 \qquad (i = 1, 2) \tag{4.3}$$

are satisfied we shall have probable asymptotic stability of the process $\varphi \equiv 0$.

Let

$$\varphi(l) = 0, \qquad a_1(x) = a_1 = \text{const} \geqslant 0, \qquad a_2(x) = a_2 = \text{const} \geqslant 0$$

$$Q_1(x) = Q_1 = \text{const} > 0, \qquad Q_2(x) = Q_2 = \text{const} > 0$$

Then, conditions (4.3) are rewritten as

$$[-b_1(x) + {}^1/_2\alpha_{12}]\, Q_1 - {}^1/_2\alpha_{12} Q_2 = 1 > 0$$

$$[-b_2(x) + {}^1/_2\alpha_{21}]\, Q_2 - {}^1/_2\alpha_{21} Q_1 = 1 > 0$$

or, since $Q_1 > 0$ and $Q_2 > 0$, we get

$$b_1(x)\, b_2(x) - {}^1/_2\, [\alpha_{12} b_2(x) + \alpha_{21} b_1(x)] > 0, \quad b_1(x) < {}^1/_2\, [\alpha_{12} + \alpha_{21}]$$

$$b_2(x) < {}^1/_2\, [\alpha_{12} + \alpha_{21}] \tag{4.4}$$

Equation (4.1) describes a stochastic process. Let us compare this process with the two deterministic processes which correspond to the realizations $u = u_1$ and $u = u_2$. For simplicity we set

$$a(x, u) \equiv 0, \qquad b_1 \equiv -{}^1/_3, \qquad b_2 \equiv {}^1/_8, \qquad \alpha_{12} \equiv \alpha_{21} \equiv {}^1/_2$$

If we consider the deterministic processes

$$d\varphi / dt = -{}^1/_3 \varphi, \qquad d\varphi / dt = {}^1/_8 \varphi$$

then the first of these is Liapunov stable while the second is unstable.

However, if we consider the stochastic process

$$d\varphi / dt = b(u)\, \varphi$$

where $b(u_1) = -{}^1/_3$, $b(u_2) = {}^1/_8$, $\alpha_{12} = \alpha_{21} = {}^1/_2$, then this process is probably asymptotically stable since conditions (4.4) are satisfied.

2. Consider Equation

$$\frac{\partial \varphi}{\partial t} = a(x, u) \frac{\partial^2 \varphi}{\partial x^2} + b(x, u) \frac{\partial \varphi}{\partial x} + c(x, u)\, \varphi \tag{4.5}$$

where the random parameter u can take two discrete values u_1 and u_2 The boundary conditions are $\varphi(0,t) = \varphi(l,t) = 0$.

Let us investigate for probable stability in the metric

$$\rho \equiv \rho_0 \equiv \left\{ \int_0^l \varphi^2 dx \right\}^{1/2}$$

Let

$$v = \frac{1}{2} \int_0^l Q(x, u)\, \varphi^2 dx, \qquad Q(x, u) \geqslant \varepsilon > 0$$

Taking the boundary conditions into account, for the mean derivative we get

$$\frac{dM_i[v]}{dt} = -\frac{1}{2} \int_0^l \left[2a(x, u_i)\, Q(x, u_i) \left(\frac{\partial \varphi}{\partial x} \right)^2 + B_i(x)\, \varphi^2 \right] dx$$

Here

$$B_i(x) = \frac{\partial b(x, u_i)\, Q(x, u_i)}{\partial x} - \frac{\partial^2 a(x, u_i)\, Q(x, u_i)}{\partial x^2} -$$

$$- 2C(x, u_i)\, Q(x, u_i) - \alpha_{ij} [Q(x, u_j) - Q(x, u_i)]$$

The inequalities

$$a(x, u_i) \geqslant 0, \qquad B_i(x) \geqslant 0 \qquad (i = 1,2) \tag{4.6}$$

give the sufficient conditions for probable stability.

3. Let the velocity profile of the fundamental motion of a liquid be rectangular, $v_0 = a(u) + b(u)y$, where u is a random parameter which may take the values u_1 and u_2 . The planar perturbed motion of the liquid is described by Equations [2]

$$\frac{\partial \omega}{\partial t} = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) - v_0 \frac{\partial \omega}{\partial x}, \quad \left(\omega = \frac{1}{2} \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) = -\frac{1}{2} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \right) \tag{4.7}$$

Here v_1, v_2 are the components of the perturbations of the velocity vector, ψ is the stream function of the perturbations. We introduce the metric

$$\rho \equiv \rho_0 \equiv \left\{\iint_\tau \omega^2 \, d\tau\right\}^{1/2}$$

and the functional

$$v = \frac{1}{2} \int_\tau Q(u)\, \omega^2 \, d\tau$$

The region τ is taken to be the rectangle $l_1 \leqslant x \leqslant l_2$, $h_1 \leqslant y \leqslant h_2$. Then for the mean derivative we obtain

$$\frac{dM_t[v]}{dt} = \nu(u_i)\, Q(u_i) \int_{l_1}^{l_2} \left[\left(\omega \frac{\partial\omega}{\partial y}\right)_{h_2} - \left(\omega \frac{\partial\omega}{\partial y}\right)_{h_2}\right] dx +$$

$$+ \nu(u_i)\, Q(u_i) \int_{h_1}^{h_2} \left[\left(\omega \frac{\partial\omega}{\partial x}\right)_{l_2} - \left(\omega \frac{\partial\omega}{\partial x}\right)_{l_1}\right] dy - \int_{h_1}^{h_2} \frac{v_0(u_i)\, Q(u_i)}{2} \left[(\omega^2)_{l_2} - (\omega^2)_{l_1}\right] dy -$$

$$- \int_\tau \left\{\nu(u_i)\, Q(u_i) \left[\left(\frac{\partial\omega}{\partial x}\right)^2 + \left(\frac{\partial\omega}{\partial y}\right)^2\right] - \alpha_{ij}\left[Q(u_j) - Q(u_i)\right]\omega^2\right\} d\tau \qquad (i, j = 1, 2;\ j \neq i)$$

Let $Q(u_j) = Q(u_i) = 1 > 0$. If the region occupied by the perturbations lies completely inside the region τ and if on the boundary and outside τ the perturbations are absent, i.e. $\omega = 0$, then

$$\frac{dM_t[v]}{dt} = -\nu(u_i) \int_\tau \left[\left(\frac{\partial\omega}{\partial x}\right)^2 + \left(\frac{\partial\omega}{\partial y}\right)^2\right] d\tau \qquad (i = 1, 2)$$

The coefficient of viscosity $\nu = \nu(u_i)$ is always a positive quantity and, therefore, $dM_t[v]/dt \leqslant 0$.

Thus, the process $\omega \equiv 0$ is probably stable.

The particular case of $v_0 \equiv 0$ corresponds to the dispersion of the vortices in an unbounded viscous liquid [3] for random changes in the coefficient of viscosity.

4. The differential equations of the perturbations of the plane-parallel isothermal motion of a gas with a fundamental velocity v_0 not dependent on x and y, can be written as

$$\frac{\partial v_1}{\partial t} = -v_0 \frac{\partial v_1}{\partial x} - \frac{1}{\rho_0'} \frac{\partial P}{\partial x} + \nu\left(\frac{4}{3} \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} \frac{1}{3} \frac{\partial^2 v_2}{\partial x \partial y}\right)$$

$$\frac{\partial v_2}{\partial t} = -v_0 \frac{\partial v_2}{\partial x} - \frac{1}{\rho_0'} \frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 v_2}{\partial x^2} + \frac{4}{3} \frac{\partial^2 v_2}{\partial y^2} + \frac{1}{3} \frac{\partial^2 v_1}{\partial x \partial y}\right)$$

$$\frac{\partial \rho'}{\partial t} = -\rho_0'\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\right) - v_0 \frac{\partial \rho'}{\partial x}, \qquad P = RT_0\rho' \qquad (4.8)$$

where v_1, v_2 are the components of the velocity perturbations, P, ρ' are, respectively, the pressure and density perturbations, v_0, ρ_0', T_0 are, respectively, the velocity, density and temperature of the unperturbed motion, not dependent on the coordinates x, y.

The quantities $v_0 = v_0(u)$, $\rho_0' = \rho_0'(u)$, $T_0 = T_0(u)$, $\nu = \nu(u)$ depend on the random parameter u which takes the values $u_1, \ldots, u_n$. Let

$$\rho \equiv \rho_0 \equiv \left\{\iint_\tau (v_1^2 + v_2^2 + \rho'^2)\, d\tau\right\}^{1/2}$$

$$v = \frac{1}{2} \int_\tau Q(u) \left(v_1^2 + v_2^2 + \frac{RT_0}{\rho_0'^2} \rho'^2\right) d\tau$$

Then, taking into account that $v_1 = v_2 = 0$ on the boundary of the region, and by setting

$$\int_{h_1}^{h_2} [(\rho^2)_{l_2} - (\rho^2)_{l_1}]\, dx, \qquad Q(u_i) = Q(u_j) = 1 > 0$$

for the mean derivative we get (4.9)

$$\frac{dM_t[v]}{dt} = -\nu(u_i) \int_\tau \left[\frac{4}{3}\left(\frac{\partial v_1}{\partial x}\right)^2 + \left(\frac{\partial v_1}{\partial y}\right)^2 + \frac{2}{3}\frac{\partial v_1}{\partial x}\frac{\partial v_2}{\partial y} + \left(\frac{\partial v_2}{\partial x}\right)^2 + \frac{4}{3}\left(\frac{\partial v_2}{\partial y}\right)^2\right] d\tau$$

When $\nu(u_i) \geqslant 0$ the stochastic process described by system (4.8) is probably stable.

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